

# INTEGRATING RATIONAL FUNCTIONS

To find  $\int \frac{N(x)}{D(x)} dx$ , where  $N(x)$  and  $D(x)$  are both polynomials, and no cancellation is possible

Some rational integrands require partial fractions, others do not, and still others are a combination of the two. The process below outlines when you should and should not use partial fractions.

If degree of  $N(x) \geq$  degree of  $D(x)$  (ie. numerator has same degree as, or higher degree than, denominator)

1. Perform polynomial long division
2. Rewrite integrand as polynomial + remainder (with degree  $<$  degree of  $D(x)$ )
3. Use processes below to find the integral of  $\frac{\text{remainder}}{D(x)}$  (ie. set  $N(x) = \text{remainder}$  and continue)

If degree of  $D(x) = 1$  (ie. denominator is linear)

1. Let  $u = D(x)$  & perform  $u$  - substitution  
(or use guess & check - antiderivative is a multiple of  $\ln|D(x)|$ )

eg. 
$$\int \frac{13}{7x-5} dx \quad \text{Let } u = 7x-5, \text{ so } \frac{du}{dx} = 7 \text{ and } dx = \frac{1}{7} du$$
$$= \int \frac{13 du}{7 u} = \frac{13}{7} \ln|u| + C = \frac{13}{7} \ln|7x-5| + C$$

If  $N(x) = k \cdot D'(x)$  (ie. numerator is constant multiple of derivative of denominator)

TYPE 1

1. Let  $u = D(x)$  & perform  $u$  - substitution  
(or use guess & check - antiderivative is a multiple of  $\ln|D(x)|$ )

eg. 
$$\int \frac{20-5x}{x^2-8x+25} dx \quad \text{Let } u = x^2-8x+25, \text{ so } \frac{du}{dx} = 2x-8 \text{ and } 20-5x = -\frac{5}{2}(2x-8)$$
$$= \int -\frac{5 du}{2 u} = -\frac{5}{2} \ln|u| + C = -\frac{5}{2} \ln|x^2-8x+25| + C$$

If degree of  $D(x) = 2$  and is irreducible

(ie. denominator is quadratic with negative discriminant, so denominator has no real roots / only complex roots)

If  $N(x) = c$  (ie. numerator is constant)

TYPE 2

1. Factor leading coefficient from  $D(x)$  (ie. so denominator starts with  $x^2$ )
2. Complete the square for  $D(x) = (x+h)^2 + a^2$
3. Factor  $a^2$  from denominator, let  $u = \frac{x+h}{a}$  & perform  $u$  - substitution

(or use  $\int \frac{a}{(x+h)^2 + a^2} dx = \tan^{-1} \frac{x+h}{a}$ )

eg. 
$$\int \frac{7}{2x^2-16x+50} dx = \frac{7}{2} \int \frac{1}{x^2-8x+25} dx = \frac{7}{2} \frac{1}{3} \int \frac{3}{(x-4)^2+3^2} dx = \frac{7}{6} \tan^{-1} \frac{x-4}{3} + C$$

**If  $N(x) = ax + b$  (ie. numerator is linear)****TYPE 3**

- Use technique similar to partial fractions shortcut to rewrite numerator as  $A \cdot D'(x) + B \cdot a$   
(ie. constant multiple of derivative of denominator +  
constant multiple of square root of “leftover” constant from completing square in denominator)  
(see Partial Fractions Decomposition handout (special note regarding Math 1B))
- Split integrand into integrand of **TYPE 1** + integrand of **TYPE 2**
- Use processes above (NOTE: no absolute values required in  $\ln(D(x))$ )

eg. 
$$\int \frac{3x-17}{x^2-8x+25} dx = \int \frac{A(2x-8)+B(3)}{(x-4)^2+3^2} dx \quad x=4: \quad 3(4)-17 = A(0)+B(3) \Rightarrow B = -\frac{5}{3}$$

*coefficient of  $x$ :  $3 = 2A \Rightarrow A = \frac{3}{2}$*

$$= \int \frac{\frac{3}{2}(2x-8) - \frac{5}{3}(3)}{(x-4)^2+3^2} dx = \frac{3}{2} \int \frac{2x-8}{x^2-8x+25} dx - \frac{5}{3} \int \frac{3}{(x-4)^2+3^2} dx$$

$$= \frac{3}{2} \ln(x^2-8x+25) - \frac{5}{3} \tan^{-1} \frac{x-4}{3} + C$$

NOTE: TYPE 3 rational functions can also be integrated using a trigonometric substitution.

eg. 
$$\int \frac{3x-17}{x^2-8x+25} dx = \int \frac{3x-17}{(x-4)^2+9} dx \quad \text{Let } x = 4 + 3 \tan \theta, \text{ so } dx = 3 \sec^2 \theta d\theta$$

$$= \int \frac{3(4+3 \tan \theta) - 17}{(3 \tan \theta)^2 + 9} 3 \sec^2 \theta d\theta = \int \frac{9 \tan \theta - 5}{3} d\theta = 3 \ln |\sec \theta| - \frac{5}{3} \theta + C$$

$$= 3 \ln \sqrt{(x-4)^2+9} - \frac{5}{3} \tan^{-1} \frac{x-4}{3} + C = \frac{3}{2} \ln((x-4)^2+9) - \frac{5}{3} \tan^{-1} \frac{x-4}{3} + C$$

**All other cases require partial fractions**

- Perform partial fractions decomposition (see handout)  
NOTE: for irreducible quadratic denominators  $d(x) = ax^2 + bx + c$   
(or powers of these factors, ie.  $[d(x)]^n$  or  $(ax^2 + bx + c)^n$ )  
write numerator in  $A \cdot D'(x) + B \cdot a$  form for **TYPE 3** to save work later on
- For all partial fractions with linear and irreducible quadratic denominators:  
Use processes above
- For all partial fractions with denominator  $[d(x)]^n = (ax + b)^n$  (ie. power of linear factor):  
Let  $u = ax + b$  & perform  $u$  - substitution  
(or use guess & check – antiderivative is a multiple of  $\frac{1}{(ax + b)^{n-1}}$ )
- For all partial fractions with denominator  $[d(x)]^n = (ax^2 + bx + c)^n$  (ie. power of irreducible quadratic factor):  
Split integrand into integrand with numerator  $A \cdot d'(x)$  + integrand with numerator  $B$   
For first integrand:  
Let  $u = ax^2 + bx + c$  & perform  $u$  - substitution  
(or use guess & check – antiderivative is a multiple of  $\frac{1}{(ax^2 + bx + c)^{n-1}}$ )  
For second integrand:  
Factor leading coefficient from  $ax^2 + bx + c$  (ie. so irreducible quadratic starts with  $x^2$ )  
Complete the square for  $x^2 + Bx + C = (x + h)^2 + k^2$   
Let  $x + h = k \tan \theta$  & perform trigonometric substitution  
**NOTE: This is the hardest type – there will be no required problems of this type on tests**

Practice against the following examples:

**TYPE 1**

$$\int \frac{7}{3x+8} dx = \frac{7}{3} \ln|3x+8| + C$$

**TYPE 1**

$$\int \frac{6-9x}{3x^2-4x-4} dx = -\frac{3}{2} \ln|3x^2-4x-4| + C$$

**TYPE 2**

$$\int \frac{5}{4x^2+24x+52} dx = \frac{5}{4} \int \frac{1}{x^2+6x+13} dx = \frac{5}{4} \frac{1}{2} \int \frac{2}{(x+3)^2+2^2} dx = \frac{5}{8} \tan^{-1} \frac{x+3}{2} + C$$

**TYPE 3**

$$\begin{aligned} \int \frac{5x-7}{x^2+8x+25} dx &= \int \frac{\frac{5}{2}(2x+8) - 9(3)}{(x+4)^2+3^2} dx = \frac{5}{2} \int \frac{2x+8}{(x+4)^2+3^2} dx - 9 \int \frac{3}{(x+4)^2+3^2} dx \\ &= \frac{5}{2} \ln(x^2+8x+25) - 9 \tan^{-1} \frac{x+4}{3} + C \end{aligned}$$

**MIXED**

$$\begin{aligned} \int \frac{-20x-12}{(x+1)^2(x^2+4x+7)} dx &= \int \left( \frac{-6}{x+1} + \frac{2}{(x+1)^2} + \frac{3(2x+4) + \frac{4}{\sqrt{3}}(\sqrt{3})}{(x+2)^2+(\sqrt{3})^2} \right) dx \\ &= \int \frac{-6}{x+1} dx + \int \frac{2}{(x+1)^2} dx + 3 \int \frac{2x+4}{x^2+4x+7} dx + \frac{4}{\sqrt{3}} \int \frac{\sqrt{3}}{(x+2)^2+(\sqrt{3})^2} dx \\ &= -6 \ln|x+1| - \frac{2}{x+1} + 3 \ln(x^2+4x+7) + \frac{4}{\sqrt{3}} \tan^{-1} \frac{x+2}{\sqrt{3}} + C \end{aligned}$$

**MIXED**

$$\begin{aligned} \int \frac{3x^3-11x^2-49x}{x^2-5x-6} dx &= \int \left( 3x+4 + \frac{-11x+24}{(x-6)(x+1)} \right) dx = \int \left( 3x+4 - \frac{6}{x-6} - \frac{5}{x+1} \right) dx \\ &= \frac{3}{2} x^2 + 4x - 6 \ln|x-6| - 5 \ln|x+1| + C \end{aligned}$$